

Title	The Chow rings of the algebraic groups $E_6$ , $E_7$ , and $E_8$ (Cohomology Theory of Finite Groups and Related Topics)
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Citation	数理解析研究所講究録 (2008), 1581: 92-102
Issue Date	2008-02
URL	<a href="http://hdl.handle.net/2433/81426">http://hdl.handle.net/2433/81426</a>
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Type	Departmental Bulletin Paper
Textversion	publisher

# The Chow rings of the algebraic groups $E_6$ , $E_7$ , and $E_8$

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## 1 Introduction

Let  $G$  be a simply connected, simple algebraic group over the complex numbers  $\mathbb{C}$ ,  $B$  a Borel subgroup and  $H$  a maximal torus contained in  $B$ . Denote by  $\hat{H}$  the character group of  $H$ . By taking the first Chern class of the homogeneous line bundle  $L_\chi$  over the flag variety  $G/B$  associated to each character  $\chi$ , we define the *characteristic homomorphism* for  $G$ ,

$$c_G : S(\hat{H}) \longrightarrow A(G/B), \quad (1)$$

where  $S(\hat{H})$  is the symmetric algebra of  $\hat{H}$  and  $A(G/B) = \bigoplus_{i \geq 0} A^i(G/B)$  is the Chow ring of the algebraic variety  $G/B$ .

According to Grothendieck's remark ([6], p.21, REMARQUES 2°), the Chow ring  $A(G)$  of  $G$  is obtained as the quotient of  $A(G/B)$  by the ideal

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Cohomology Theory of Finite Groups and Related Topics, August 27-31, 2007.

2000 *Mathematics Subject Classification*. Primary 14C15; Secondary 14M15.

*Key words and phrases*. Chow rings, algebraic groups, Schubert calculus, flag varieties.

\* Partially supported by the Grant-in-Aid for Scientific Research (C), Japan Society of the Promotion of Science.

generated by the image of  $\hat{H}$  under  $c_G$ . Following this remark,  $A(G)$  for  $G = \mathrm{SO}(n), \mathrm{Spin}(n), G_2$ , and  $F_4$  were computed by R. Marlin [8]. So the remaining simply connected simple groups are  $E_6, E_7$ , and  $E_8$ .

**Problem 1.1** *Determine the Chow rings of  $E_6, E_7$ , and  $E_8$ .*

## 2 Computations of $A(G/B)$

In order to determine the Chow ring  $A(G)$  of  $G$  following Grothendieck's remark, we have to compute the Chow ring  $A(G/B)$  of the corresponding flag variety  $G/B$ . As for the Chow rings of flag varieties, the following fact is known.

**Fact 2.1** *The Chow ring  $A(G/B)$  is isomorphic to the integral cohomology ring  $H^*(G/B; \mathbb{Z})$  via the cycle map.*

In what follows, we consider the integral cohomology ring  $H^*(G/B; \mathbb{Z})$ . As is well known, there are two different ways of describing the cohomology of  $G/B$ . Namely, the *Borel presentation* and the *Schubert presentation*, which we now recall.

### Borel presentation

Let  $K$  be a maximal compact subgroup of  $G$  and  $T = K \cap H$  a maximal torus of  $K$ . Then we have the diffeomorphism  $G/B \cong K/T$  by the Iwasawa decomposition of  $G$ . According to Borel, there exists a fibration

$$K/T \xrightarrow{\iota} BT \xrightarrow{\rho} BK,$$

where  $BT$  (resp.  $BK$ ) denotes the classifying space of  $T$  (resp.  $K$ ). The induced homomorphism in cohomology,

$$c = \iota^* : H^*(BT; \mathbb{Z}) \longrightarrow H^*(K/T; \mathbb{Z}) \quad (2)$$

is called *Borel's characteristic homomorphism* and can be identified with the characteristic homomorphism (1). The Weyl group  $W$  of  $K$  acts naturally on  $T$ , hence on  $H^2(BT; \mathbb{Z})$ . We extend this action of  $W$  to the whole  $H^*(BT; \mathbb{Z})$  and also to  $H^*(BT; \mathbb{F}) = H^*(BT; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$ , where  $\mathbb{F}$  is any field. Then one of Borel's results can be stated as follows.

**Theorem 2.2** *Let  $\mathbb{F}$  be a field of characteristic zero. Then Borel's characteristic homomorphism induces an isomorphism,*

$$\bar{c} : H^*(BT; \mathbb{F}) / (H^+(BT; \mathbb{F})^W) \longrightarrow H^*(K/T; \mathbb{F}),$$

where  $(H^+(BT; \mathbb{F})^W)$  is the ideal of  $H^*(BT; \mathbb{F})$  generated by the  $W$ -invariants of positive degrees.

In particular, one can reduce the computation of the rational cohomology ring  $H^*(K/T; \mathbb{Q})$  to that of the ring of invariants  $H^*(BT; \mathbb{Q})^W$ . In order to determine the integral cohomology ring  $H^*(K/T; \mathbb{Z})$ , we need further considerations. General description of  $H^*(K/T; \mathbb{Z})$  by a minimal system of generators and relations was given by H. Toda [12]. Up to now, the following results have been available.

$H^*(SU(n+1)/T; \mathbb{Z})$	...	Borel (1953),
$H^*(SO(2n+1)/T; \mathbb{Z})$	...	Toda-Watanabe (1974),
$H^*(Sp(n)/T; \mathbb{Z})$	...	Borel (1953),
$H^*(SO(2n)/T; \mathbb{Z})$	...	Toda-Watanabe (1974),
$H^*(G_2/T; \mathbb{Z})$	...	Bott-Samelson (1955),
$H^*(F_4/T; \mathbb{Z})$	...	Toda-Watanabe (1974),
$H^*(E_6/T; \mathbb{Z})$	...	Toda-Watanabe (1974),
$H^*(E_7/T; \mathbb{Z})$	...	Nakagawa (2001),
$H^*(E_8/T; \mathbb{Z})$	...	Nakagawa (2007).

**Remark 2.3** *In the Borel presentation, the ring structure of  $H^*(K/T; \mathbb{Z})$  is relatively easy to obtain. However, the ring generators have little “geometric meaning” in this presentation.*

### Schubert presentation

As is well known,  $G$  has the Bruhat decomposition,

$$G = \coprod_{w \in W} B\dot{w}B,$$

where  $\dot{w}$  denotes any representative of  $w \in W$ . It induces a cell decomposition,

$$G/B = \coprod_{w \in W} B\dot{w}B/B,$$

where  $X_w^\circ = BwB/B \cong \mathbb{C}^{l(w)}$  is called the *Schubert cell*. Here  $l(w)$  is the length of the element  $w \in W$ . The *Schubert variety*  $X_w$  is defined to be the closure of  $X_w^\circ$ . Denote by  $[X_w] \in H_{2l(w)}(G/B; \mathbb{Z})$  the image of the fundamental class  $[X_w] \in H_{2l(w)}(X_w; \mathbb{Z})$  under the induced homomorphism by the inclusion  $X_w \hookrightarrow G/B$ . We define a cohomology class  $Z_w \in H^{2l(w)}(G/B; \mathbb{Z})$  as the Poincaré dual of  $[X_{w_0w}]$ , where  $w_0$  is the longest element of  $W$ . We call  $Z_w$  the *Schubert class*. Then we have

**Fact 2.4** *The Schubert classes  $\{Z_w\}_{w \in W}$  form an additive basis for  $H^*(G/B; \mathbb{Z})$ . We refer to  $\{Z_w\}_{w \in W}$  as the Schubert basis.*

**Remark 2.5** *In the Schubert presentation, the Schubert classes correspond to the geometric objects -the Schubert varieties. However, the multiplicative structure among them is highly complicated.*

Now we consider the following problem.

**Problem 2.6** *Establish a connection between the Borel presentation and the Schubert presentation.*

Our main tool is the *divided difference operators* introduced independently by Bernstein-Gelfand-Gelfand [1] and Demazure [5].

### Divided difference operators

First we need some notation.

$\Delta$ : the root system of  $K$  with respect to  $T$ ;

$\Delta^+$ : a set of positive roots;

$\Pi$ : the system of simple roots;

$s_\alpha$ : the reflection corresponding to the simple root  $\alpha \in \Pi$ .

**Definition 2.7** (i) *For each  $\alpha \in \Delta$ , the operator*

$$\Delta_\alpha : H^*(BT; \mathbb{Z}) \longrightarrow H^*(BT; \mathbb{Z})$$

*is defined as*

$$\Delta_\alpha(u) = \frac{u - s_\alpha(u)}{\alpha} \quad \text{for } u \in H^*(BT; \mathbb{Z}).$$

(ii) For  $w \in W$ , the operator  $\Delta_w$  is defined as

$$\Delta_w = \Delta_{\alpha_1} \circ \Delta_{\alpha_2} \circ \cdots \circ \Delta_{\alpha_k},$$

where  $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$  ( $\alpha_i \in \Pi$ ) is any reduced decomposition of  $w$ .

One can show that the definition is well defined, i.e., independent of the choice of a reduced decomposition of  $w$ . Then Borel's characteristic homomorphism (2) can be described by the divided difference operators.

**Theorem 2.8 (Bernstein-Gelfand-Gelfand [1], Demazure [5])** For a homogeneous polynomial  $f \in H^{2k}(BT; \mathbb{Z})$ , we have

$$c(f) = \sum_{w \in W, l(w)=k} \Delta_w(f) Z_w. \quad (3)$$

In particular, for  $\alpha \in \Pi$ , we have

$$c(\omega_\alpha) = Z_{s_\alpha},$$

where  $\omega_\alpha$  denotes the fundamental weight corresponding to the simple root  $\alpha \in \Pi$ .

### 3 $H^*(E_l/T; \mathbb{Z})$ ( $l = 6, 7, 8$ )

Let  $E_l$  ( $l = 6, 7, 8$ ) be the simply connected simple complex algebraic group of exceptional type,  $E_l$  its maximal compact subgroup and  $T$  a maximal torus of  $E_l$ . According to [4], we take the simple roots  $\{\alpha_i\}_{1 \leq i \leq l}$  and denote by  $\{\omega_i\}_{1 \leq i \leq l}$  the corresponding fundamental weights. Let  $s_i$  ( $1 \leq i \leq l$ ) denote the reflection corresponding to the simple root  $\alpha_i$  ( $1 \leq i \leq l$ ). Then the Weyl group  $W(E_l)$  of  $E_l$  is generated by  $s_i$  ( $1 \leq i \leq l$ ). As usual, we regard roots and weights as elements of  $H^2(BT; \mathbb{Z})$ . Following the notation in [11], [9], and [10], we put

$$\begin{aligned} t_l &= \omega_l, \\ t_i &= s_{i+1}(t_{i+1}) \quad (2 \leq i \leq l-1), \\ t_1 &= s_1(t_2), \\ t &= \omega_2, \\ c_i &= \sigma_i(t_1, \dots, t_l) \quad (1 \leq i \leq l), \end{aligned} \quad (4)$$

where  $\sigma_i(t_1, \dots, t_l)$  denotes the  $i$ -th elementary symmetric function in the variables  $t_1, \dots, t_l$ . Then we have

$$\begin{aligned} H^*(BT; \mathbb{Z}) &= \mathbb{Z}[\omega_1, \omega_2, \dots, \omega_l] \\ &= \mathbb{Z}[t_1, t_2, \dots, t_l, t]/(c_1 - 3t). \end{aligned}$$

Since we consider the simply connected form of the groups, Borel's characteristic homomorphism restricted in degree 2 is an isomorphism:

$$c = \iota^* : H^2(BT; \mathbb{Z}) \longrightarrow H^2(E_l/T; \mathbb{Z}).$$

Under this isomorphism, we denote the images of  $t_i$  ( $1 \leq i \leq l$ ) and  $t$  by the same symbols. Thus  $H^2(E_l/T; \mathbb{Z})$  is a free  $\mathbb{Z}$ -module generated by  $t_i$  ( $1 \leq i \leq l$ ) and  $t$  with a relation  $c_1 = 3t$ .

Then the integral cohomology ring of  $E_6/T$  is given as follows.

**Theorem 3.1** ([11], Theorem B) *The integral cohomology ring of  $E_6/T$  is*

$$H^*(E_6/T; \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_6, t, \gamma_3, \gamma_4]/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{12}),$$

where

$$\begin{aligned} \rho_1 &= c_1 - 3t, \\ \rho_2 &= c_2 - 4t^2, \\ \rho_3 &= c_3 - 2\gamma_3, \\ \rho_4 &= c_4 + 2t^4 - 3\gamma_4, \\ \rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3, \\ \rho_6 &= \gamma_3^2 + 2c_6 - 3t^2\gamma_4 + t^6, \\ \rho_8 &= 3\gamma_4^2 - 6t\gamma_3\gamma_4 - 9t^2c_6 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8, \\ \rho_9 &= 2c_6\gamma_3 - 3t^3c_6, \\ \rho_{12} &= 3c_6^2 - 2\gamma_4^3 + 6t\gamma_3\gamma_4^2 + 3t^2c_6\gamma_4 + 5t^3c_6\gamma_3 - 15t^4\gamma_4^2 - 10t^6c_6 \\ &\quad + 19t^8\gamma_4 - 6t^9\gamma_3 - 2t^{12}. \end{aligned}$$

Similar presentations of  $H^*(E_l/T; \mathbb{Z})$  ( $l = 7, 8$ ) are also obtained in [9] and [10].

Now we consider the following problem.

**Problem 3.2** Find the relations between the ring generators  $\{t_1, \dots, t_l, t, \gamma_3, \gamma_4, \dots\}$  in the Borel presentation and the Schubert basis  $\{Z_w\}_{w \in W(E_l)}$  ( $l = 6, 7, 8$ ).

We will show how to do this in the case of  $E_6$ . Since  $c(\omega_i) = Z_i$  by Theorem 2.8, it follows immediately from (4) that

$$\begin{aligned}
 t_1 &= -Z_1 + Z_2, \\
 t_2 &= Z_1 + Z_2 - Z_3, \\
 t_3 &= Z_2 + Z_3 - Z_4, \\
 t_4 &= Z_4 - Z_5, \\
 t_5 &= Z_5 - Z_6, \\
 t_6 &= Z_6, \\
 t &= Z_2.
 \end{aligned} \tag{5}$$

For  $i = 3, 4$ , we can put

$$\gamma_i = \sum_{l(w)=i} a_w Z_w$$

for some integers  $a_w$ . We will determine the coefficients  $a_w$ . By Theorem 3.1, we have

$$\begin{aligned}
 2\gamma_3 &= c_3, \\
 3\gamma_4 &= c_4 + 2t^4.
 \end{aligned} \tag{6}$$

Therefore  $2\gamma_3$  and  $3\gamma_4$  are contained in the image of  $c$ . Define the polynomials of  $H^*(BT; \mathbb{Z})$  by

$$\begin{aligned}
 \delta_3 &= c_3, \\
 \delta_4 &= c_4 + 2t^4,
 \end{aligned} \tag{7}$$

so that  $c(\delta_3) = c_3 (= 2\gamma_3)$ ,  $c(\delta_4) = c_4 + 2t^4 (= 3\gamma_4)$  in  $H^*(E_6/T; \mathbb{Z})$ . We apply the divided difference operators to the polynomials  $\delta_3$  and  $\delta_4$ .

Thus we obtain



$$\begin{aligned}
c_3 &= 2Z_{342} + 4Z_{542} \\
&= 2(Z_{342} + 2Z_{542}), \\
c_4 + 2t^4 &= 3Z_{1342} + 6Z_{3542} + 6Z_{6542} \\
&= 3(Z_{1342} + 2Z_{3542} + 2Z_{6542}).
\end{aligned} \tag{8}$$

By (6) and (8), we can express  $\gamma_i$  ( $i = 3, 4$ ) in terms of Schubert classes. Since  $H^*(E_6/T; \mathbb{Z})$  is torsion free, we obtain

$$\begin{aligned}
\gamma_3 &= Z_{342} + 2Z_{542}, \\
\gamma_4 &= Z_{1342} + 2Z_{3542} + 2Z_{6542}.
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
Z_{342} &= -\gamma_3 + 2t^3, \\
Z_{542} &= \gamma_3 - t^3, \\
Z_{1342} &= \gamma_4 - 2t\gamma_3 + 2t^4, \\
Z_{3542} &= -\gamma_4 + t\gamma_3, \\
Z_{6542} &= \gamma_4 - t^4.
\end{aligned} \tag{9}$$

## 4 Computations of $A(G)$

In this section, we determine the Chow rings of the exceptional groups  $E_6, E_7$ , and  $E_8$ . Since we have the following commutative diagram,

$$\begin{array}{ccc}
S(\hat{H}) & \xrightarrow{c_G} & A(G/B) \\
\cong \downarrow & & \downarrow \cong \\
H^*(BT; \mathbb{Z}) & \xrightarrow{c} & H^*(G/B; \mathbb{Z}),
\end{array}$$

we have

$$\begin{aligned}
A(G) &= A(G/B)/(c_G(\hat{H})) \\
&= H^*(G/B; \mathbb{Z})/(c(H^2(BT; \mathbb{Z}))) \\
&= H^*(G/B; \mathbb{Z})/(H^2(G/B; \mathbb{Z})) \\
&= H^*(K/T; \mathbb{Z})/(H^2(K/T; \mathbb{Z})).
\end{aligned}$$

Therefore we have only to compute the quotient ring of  $H^*(K/T; \mathbb{Z})$  by the ideal generated by  $H^2(K/T; \mathbb{Z})$ . We will show how to do this for the case of  $E_6$ . By Theorem 3.1 and (9), we compute

$$\begin{aligned}
H^*(E_6/T; \mathbb{Z})/(H^2(E_6/T; \mathbb{Z})) &= H^*(E_6/T; \mathbb{Z})/(t_1, \dots, t_6, t) \\
&= \mathbb{Z}[\gamma_3, \gamma_4]/(2\gamma_3, 3\gamma_4, \gamma_3^2, \gamma_4^3) \\
&= \mathbb{Z}[Z_{542}, Z_{6542}]/(2Z_{542}, 3Z_{6542}, Z_{542}^2, Z_{6542}^3).
\end{aligned}$$

In this way, we can compute the Chow rings of  $E_l$  ( $l = 6, 7, 8$ ). Let  $T_G : A(G/B) \rightarrow A(G)$  denote the natural projection and  $w_0$  the longest element of the Weyl group  $W(E_l)$  ( $l = 6, 7, 8$ ). Then we have the following main result.

**Theorem 4.1** (i) *The Chow ring of  $E_6$  is*

$$A(E_6) = \mathbb{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3),$$

where  $X_3 = T_{E_6}(X_{w_0 s_5 s_4 s_2})$  and  $X_4 = T_{E_6}(X_{w_0 s_6 s_5 s_4 s_2})$ .

(ii) *The Chow ring of  $E_7$  is*

$$\begin{aligned}
A(E_7) &= \mathbb{Z}[X_3, X_4, X_5, X_9] \\
&\quad / (2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2),
\end{aligned}$$

where  $X_3 = T_{E_7}(X_{w_0 s_5 s_4 s_2})$ ,  $X_4 = T_{E_7}(X_{w_0 s_6 s_5 s_4 s_2})$ ,  $X_5 = T_{E_7}(X_{w_0 s_7 s_6 s_5 s_4 s_2})$ ,  $X_9 = T_{E_7}(X_{w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2})$ .

(iii) *The Chow ring of  $E_8$  is*

$$\begin{aligned}
A(E_8) &= \mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \\
&\quad / \left( \begin{array}{l} 2X_3, 3X_4, 2X_5, 5X_6, 2X_9, X_5^2 - 3X_{10}, X_4^3, \\ 2X_{15}, X_9^2, 3X_{10}^2, X_3^8, X_{15}^2 + X_{10}^3 + 2X_6^5 \end{array} \right),
\end{aligned}$$

where  $X_i = T_{E_8}(\gamma_i)$  ( $i = 3, 4, 5, 6, 9, 10, 15$ ).

**Remark 4.2** (i) *The result of  $E_8$  is not satisfactory. We determined merely the ring structure of  $A(E_8)$ . At present, we are not able to express the ring generators of  $H^*(E_8/T; \mathbb{Z})$  in terms of Schubert classes.*

(ii) *For details on the computations for  $E_6$  and  $E_7$ , see [7].*

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